An inequality from IMO 2009 Shortlist from Estonia

https://www.linkedin.com/feed/update/urn:li:activity:6806242615911182336

Let a,b,c be positive real numbers such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=a+b+c$.

Prove that $\sum \frac{1}{(2a+b+c)^2} \leq \frac{3}{16}$.

Solution by Arkady Alt, **San Jose**,**California**, **USA.**

Since $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=a+b+c$ then $\sum \frac{1}{(2a+b+c)^2} \leq \frac{3}{16} \Leftrightarrow$

(**1**) $\sum \frac{1}{(2a+b+c)^2} \cdot \sum a \leq \frac{3}{16} \cdot \sum \frac{1}{a}$.

Homogeneous inequality (**1**) after normalization by $a+b+c=1$ becomes

$$\sum \frac{1}{(1+a)^2} \leq \frac{3}{16} \sum \frac{1}{a}.$$

Noting that by AM-GM Inequality $1+a=3\cdot\frac{1}{3}+a \geq 4\left(\frac{a}{3^3}\right)^{1/4} \Leftrightarrow (1+a)^2 \geq \frac{16\sqrt{a}}{3\sqrt{3}}$

we obtain $\sum \frac{1}{(1+a)^2} \leq \frac{3\sqrt{3}}{16} \sum \frac{1}{\sqrt{a}}$.

Thus, remains to prove inequality $\sqrt{3} \sum \frac{1}{\sqrt{a}} \leq \sum \frac{1}{a}$.

Since $\sum \frac{1}{a} \cdot \sum a \geq 9 \Leftrightarrow \sum \frac{1}{a} \geq 9 \Leftrightarrow \sqrt{\sum \frac{1}{a}} \geq 3 \Leftrightarrow \sum \frac{1}{a} \geq 3\sqrt{\sum \frac{1}{a}}$

and $\sqrt{3\cdot\sum \frac{1}{a}} \geq \sum \frac{1}{\sqrt{a}} \Leftrightarrow 3\sqrt{\sum \frac{1}{a}} \geq \sqrt{3} \sum \frac{1}{\sqrt{a}}$ (obtained by replacing (x,y,z) in $\sqrt{3(x^2+y^2+z^2)} \geq x+y+z, x,y,z>0$) then

$$\sum \frac{1}{a} \geq 3\sqrt{\sum \frac{1}{a}} \geq \sqrt{3} \sum \frac{1}{\sqrt{a}}.$$